MA1025 Exam # 2

Th. September 14th, 2006 **Name** ______________________________

Instructor: Ralucca Gera

Show all necessary work in each problem to receive credit.

1. (10 points) Solve either (a) or (b).
(a) A relation R is defined on the set $A = \{1, 2, 3\}$, by $R = \{(1, 2), (2, 1)\}$. Prove that either R is an equivalence relation, or explain why it is not an equivalence relation.
(b) The empty relation $R = \emptyset$ is defined on the set $A = \{1, 2, 3\}$. Prove that either R is an equivalence relation, or explain why it is not an equivalence relation.

(a) or (b) **Solution**: R is not an equivalence relation since it is not reflexive (for example $(1, 1) \notin R$).

2. (10 points) A relation R is defined on the set $\mathbb{Z}$, by xRy if $x - y$ is even. Prove that either R is transitive, or explain why it is not transitive.
Proof: R is transitive: Let $x, y, z \in \mathbb{Z}$ such that xRy and yRz. Then $x - y = 2k$ and $y - z = 2\ell$, for some $k, \ell \in \mathbb{Z}$. Then $x - z = (x - y) + (y - z) = 2k + 2\ell = 2(k + \ell)$. (You could also solve for x and z, and then compute $x - z$). Since $k + \ell \in \mathbf{Z}$, it follows that $x - z$ is even, and so xRz.

3. (10 points) A function $f : \mathbb{R}-\{-7\} \to \mathbb{R}-\{1\}$ is defined by $f(x) = \frac{x}{x+7}$. Assume that f is bijective. Find the inverse function $f^{-1}(x)$ (i.e. give the domain, range and formula for f^{-1}).

Proof: We solve $f(x) = y$ for x in order to get the inverse:

$$\begin{aligned} f(x) &= y \\ \frac{x}{x+7} &= y \\ x &= xy + 7y \\ x - xy &= 7y \\ x(1-y) &= 7y \\ x &= \frac{7y}{1-y}, (y \neq 1 \; as \; y \in \mathbb{R} - \{1\}). \end{aligned}$$

Thus, $f^{-1} : \mathbb{R} - \{-1\} \to \mathbb{R} - \{-7\}$ and $f^{-1}(x) = \frac{7x}{1-x}$.

Note that $1 - x \neq 0$ since $x \in \mathbb{R} - \{-1\}$. Also, $\frac{7x}{1-x} \neq 7$, since if it did, then $7x = 7(1-x)$ which simplifies to $0 = 7$ wich is a contradiction.

4. (10 points) A function $f : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ is defined by $f(x) = \frac{3}{-2x}$.

a) Prove or disprove that f is one to one (injective). (5 points)

Proof: For $a, b \in \mathbb{R} - \{0\}$, let $f(a) = f(b)$ then

$$\begin{aligned} \frac{3}{-2a} &= \frac{3}{-2b} \\ 3(-2b) &= 3(-2a) \\ -6b &= -6a \\ a &= b. \end{aligned}$$

Thus f is one to one. ■

b) Prove or disprove that f is onto (surjective). (5 points)

Proof: For $y \in \mathbb{R} - \{0\}$, let $x = \frac{3}{-2y}$ $(y \neq 0)$. Observe $\frac{3}{-2x} \in \mathbb{R} - \{0\}$ (as $\frac{3}{-2x} \neq 0$). Then

$f(x) = f(\frac{3}{-2y}) = \frac{3}{-2(\frac{3}{-2y})} = \frac{3y}{3} = y$, so f is onto. ■

5. (10 points) Suppose $|A| = 4$ and $|B| = 6$.
(a) Find the number of functions $f : A \to B$. (5 points)

Solution: Since each element of A can be mapped to any element of B, there are exactly 6 choices for each element of A to get mapped. Thus there are $6^4 = 1296$ function from A to B.

(b) Find the number of $1-1$ functions $f : A \to B$. (5 points)

Solution: Since ach element of A needs to get mapped to a unique element of B, we have exactly 6 choices for the first element, five choices for the next element, 4 choices for the third element and two choices for the last element of the set A. Thus the answer is $6 \cdot 5 \cdot 4 \cdot 3 = 360$.

6. (10 points) A committee of six is to be selected from 11 men and 8 women. In how many ways can this be done if
a) There are no restrictions?
Solution: $\binom{19}{6}$

b) There must be exactly three men and three women?
Solution: $\binom{11}{3} \cdot \binom{8}{3}$

7. (10 points) How many bitstrings of length 12 begin with 11 or end with 10?
Solution: There are 2^{10} bittrings that begin with 11, also 2^{10} that end with 10, and 2^8 that begin with 11 and end with 10 that got counted twice. Thus, we have a total of $2^{10} + 2^{10} - 2^8 = 1792$.

8. (10 points) Find the **term** that contains x^3 in the expansion of $(5x - 2y)^{10}$.
Solution: $\binom{10}{7}(5x)^3(-2y)^7 = -1,920,000x^3y^7$

9. (10 points) Use a combinatorial proof to show that

$$\binom{3n}{3} = \binom{2n}{3} + \binom{n}{3} + n\binom{2n}{2} + 2n\binom{n}{2}.$$

Proof: The number on the left hand side is the number of 3-subsets of a $3n$-set. For the right hand sie, let the $3n$-set contain say, $2n$ red and n blue elements. There are $\binom{2n}{3}$ red 3-subsets, $\binom{n}{3}$ blue 3-subsets, $\binom{2n}{2}\binom{n}{1} = n\binom{2n}{2}$ 2-red and 1-blue subsets, and $\binom{2n}{1}\binom{n}{2} = 2n\binom{n}{2}$ 1-red and 2-blue subsets. We thus have a total of $\binom{2n}{3} + \binom{n}{3} + n\binom{2n}{2} + 2n\binom{n}{2}$, giving the above relation. ∎

MA1025 Exam # 2, Take home part

DUE Th. September 14th, 2006 at 9am Name ______________________

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Show all necessary work in each problem to receive credit. You may only use your formula sheet and calculator. Time limit: 30 min.

10. (10 points) Use the principle of mathematical induction to prove that for all $n \in \mathbb{N}, \displaystyle\sum_{i=1}^{n}(i)(i!) = (n+1)! - 1.$

Proof: We prove by induction that $S_n : \displaystyle\sum_{i=1}^{n}(i!)(i) = (n+1)! - 1$ is true for all natural numbers n.

The statement $S_1 : 1! \cdot 1 = 2! - 1$ is true since $1 = 2 - 1$.

Assume that $S_k : 1 \sum_{i=1}^{k}(i!)(i) = (k+1)! - 1$ is true and prove that $S_{k+1} : \sum_{i=1}^{k+1}(i!)(i) = (k+1)! - 1$ is true. Observe that

$$
\begin{aligned}
\sum_{i=1}^{k+1}(i!)(i) &= 1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \ldots + (k+1)! \cdot (k+1) \\
&= 1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \ldots + \ldots (k)! \cdot (k) + (k+1)! \cdot (k+1) \\
&= (k+1)! - 1 + (k+1)! \cdot (k+1) \\
&= (k+1)!(1 + k + 1) - 1 \\
&= (k+1)!(k+2) - 1 \\
&= (k+2)! - 1
\end{aligned}
$$

Therefore by the Principle of Math Induction S_n is true for all natural numbers n. ■